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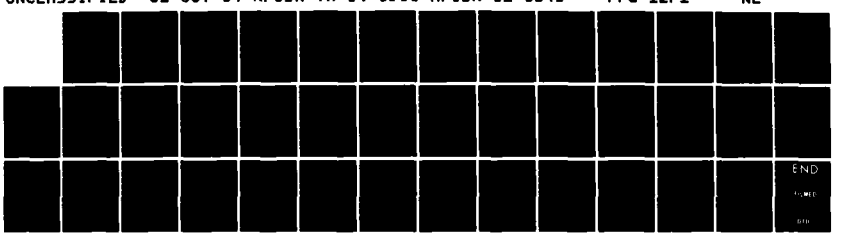

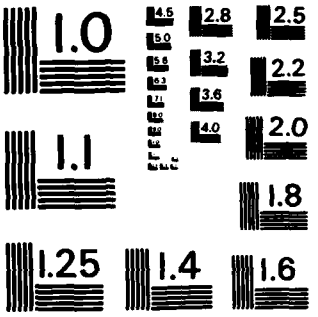

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Methods were investigated for evaluating cumulative distributions of random variables by numerical quadrature of contour integrals in the complex plane involving their moment- or probability-generating functions. These methods have been applied to the sum of M outputs of a linear rectifier whose input is Gaussian noise plus a narrowband signal, to the average power of a Gaussian random process, to the number of photoelectrons counted from a photoelectric detector onto which both a coherent light signal and incoherent background are incident, to the output of an avalanche photodiode, and to the statistics figuring in Wilcoxon rank tests. Radar detectors estimating the ambient noise level from censored samples have also been anal[illegible]

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Calculation of Cumulative Distributions and Detection Probabilities in Communications and Optics

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Statement of Work

Random variables arising in communications, radar, photodetection, and signal processing often have moment generating functions, characteristic functions, or probability generating functions for which formal mathematical expressions are readily derived theoretically. Determining the probability distributions of these variables from generating functions analytically is seldom possible. The cumulative distribution of such a variable, important because it provides false-alarm, detection, and error probabilities, can be expresssed as a contour integral in the complex plane whose integrand involves the generating function. It has been found that the distribution can be accurately computed numerically if the contour is taken through a saddlepoint of the entire integrand. We propose a thorough study of the potentialities of this method of saddlepoint integration for distributions of both continuous and discrete random variables of importance in the above-mentioned fields. When the random variable results from a quadratic functional of a Gaussian random process, the generating function involves the Fredholm determinant of the autocovariance function of the process. Its calculation for points on the contour of integration by integrating vector Riccati equations will be investigated for processes that can be modeled as the output of a linear system driven by white noise. A coherent signal component of the input process can also be handled by this analysis. Acceleration of the convergence of the numerical integration by optimum choice of the contour will be examined.

Particular applications treated in the proposal are to the detectability of fading radar echoes, the distribution of the average power of a Gaussian random process, the performance of the optimum and threshold detectors of a narrow-band Gaussian stochastic signal in white noise, the distribution of the number of photoelectrons counted when a mixture of coherent and incoherent light falls on an emissive surface, and the distribution of the number of output electrons from an avalanche photodiode. Still other applications will be considered if time and resources permit.

1. Computing Cumulative Distributions from Moments

This research is centered on the problem of computing the distribution of a random variable V when one knows its moments

$$\mu_n = E(V^n), \qquad n = 0, 1, 2, \ldots . \tag{1.1}$$

From these one forms the moment-generating function (m.g.f.)

$$h(s) = \sum_{n=0}^{\infty} \mu_n (-s)^n/n! = 1 + \sum_{n=1}^{\infty} a_n (-s)^n$$

$$= E(e^{-sV}) = \int_{-\infty}^{\infty} p(V) e^{-sV}\, dV, \tag{1.2}$$

which is simply the Laplace transform of the probability density function (p.d.f.) p(V) of the random variable V. It is assumed that h(s) is regular at least in a vertical "convergence strip" parallel to and containing the Im s-axis of the complex s-plane.

The inverse transform to (1.2) is

$$p(V) = \int_{c-i\infty}^{c+i\infty} h(s) e^{sV}\, ds/2\pi i, \tag{1.3}$$

where c lies in the convergence strip. Our concern is primarily with determining the cumulative distribution function, which is useful, for instance, in specifying false-alarm and detection probabilities in radar and error probabilities in communication systems. Integrating (1.3), one writes this as

$$q^{-}(V) = \int_{-\infty}^{V} p(V')dV' = \int_{c-i\infty}^{c+i\infty} s^{-1} h(s) e^{sV}\, ds/2\pi i, \qquad c > 0, \tag{1.4}$$

The complementary cumulative distribution is

$$q^{+}(V) = \int_{V}^{\infty} p(V')dV' = -\int_{c-i\infty}^{c+i\infty} s^{-1}\, h(s)e^{sV}\, ds/2\pi i, \qquad c > 0. \quad (1.5)$$

These integrals can seldom be worked out in closed form, and one must resort to numerical methods.

The most common procedure is to utilize the Edgeworth series. One defines the cumulant expansion by

$$\ln\, h(s) = \sum_{n=1}^{\infty} b_n(-s)^n = -\bar{V}s + \tfrac{1}{2}\,\sigma^2 s^2 + r(s),$$

$$r(s) = \sum_{n=3}^{\infty} \kappa_n(-s)^n/n!, \qquad (1.6)$$

where $\bar{V} = E(V)$, $\sigma^2 = \mathrm{Var}\ V$, and κ_n is the n-th cumulant of V. When the m.g.f. h(s) is not given in closed form, the coefficients $b_n = \kappa_n/n!$ of this series can be obtained from the coefficients $a_n = \mu_n/n!$ of (1.2) by the recurrent relations

$$b_{n+1} = a_{n+1} - \frac{1}{n+1}\sum_{r=1}^{n} r b_r a_{n+1-r}, \qquad n > 1, \qquad (1.7)$$

starting with $b_1 = a_1 = \bar{V}$.

One then expands

$$h(s) = \exp(-\bar{V}s + \tfrac{1}{2}\,\sigma^2 s^2)\{1 + \sum_{k=1}^{\infty} \frac{[r(s)]^k}{k!}\}, \qquad (1.8)$$

writes out the terms in the brackets as a series in powers of (-s), substitutes into (1.3), and integrates term by term to obtain a series for the density

function p(V). This series is in turn integrated term by term to produce, after appropriate rearrangement, the well-known Edgeworth series. It is an "asymptotic" series in the sense that although its terms at first decrease in magnitude, the series diverges when carried beyond a certain point. Those terms, proportional to derivatives of the error function, are oscillatory functions of their argument $(V-\bar{V})/\sigma$, and the terms of high order attain large positive and negative values. Computation with the series is difficult to manage, therefore, and the results are unreliable in the tails of the distribution where $|V - \bar{V}| \gg \sigma$.

In our approach one avoids the expansion in (1.8), which is the cause of all the trouble, and simply integrates (1.4) or (1.5) numerically. In order for the numerical integration to require as few steps as possible, one would like to integrate along the path of steepest descent of the integrand.[1] This path crosses the Re s-axis at a saddlepoint s_0 of the integrand, which is a root of the equation

$$\Phi'(s) = \frac{d\Phi(s)}{ds} = \frac{d}{ds} \ln h(s) + V - \frac{1}{s} = 0, \qquad (1.9)$$

where

$$\Phi(s) = \ln h(s) + sV - \ln (\pm s) \qquad (1.10)$$

is the "phase" of the integrand in (1.4) or (1.5). [$(\pm s) = +s$ for $q^-(V)$ in (1.4); $(\pm s) = -s$ for $q^+(V)$ in (1.5).] Along the path of steepest descent, which may have several branches, Im $\Phi(s)$ is constant. On the principal branch crossing the Re s-axis at $s = s_0$, Im $\Phi(s) = 0$. In many problems this is the only branch.

Because computing the path of steepest descent would be time-consuming, one approximates it. The simplest approximation is a straight vertical line through the saddlepoint s_0; that is, one takes $c = s_0$ in (1.4) or (1.5). The number of

steps of numerical integration can often be significantly reduced, however, by choosing instead a parabolic contour, on which

$$x = s_0 + \frac{1}{2}\kappa y^2, \qquad s = x + iy,$$

with the curvature κ approximating that of the path of steepest descent.[2] The integrals to be evaluated can then be put into the form

$$q^{\pm}(V) = \int_0^{\infty} \mathrm{Re}\,[\tilde{h}(s)e^{sV}(\mp s)^{-1}(1 - i\kappa y)]dy/\pi,$$

$$s = s_0 + \frac{1}{2}\kappa y^2 + iy, \tag{1.11}$$

with s_0 the appropriate root of (1.9). For $V \gg \bar{V} = E(V)$, it is most accurate to compute $q^{+}(V)$, and for that integration $s_0 < 0$; for $V \ll \bar{V}$ one computes $q^{-}(V)$, for which $s_0 > 0$. For V near $\bar{V}$ it does not matter which one evaluates.

One solves (1.9) for the saddlepoint most expeditiously by Newton's method, replacing a trial value of s_0 at each stage by

$$s_0 \leftarrow s_0 - \frac{\Phi'(s_0)}{\Phi''(s_0)}. \tag{1.12}$$

Because of the concavity of $\Phi(s)$ for $s = x$ on the real axis, this method converges rapidly. If it is inconvenient to calculate the second derivative $\Phi''(s)$, the secant method can be used instead to solve $\Phi'(s) = 0$, $s = s_0$. If even the first derivative is inconvenient to calculate, one can solve the equation

$$\mathrm{Im}\,\Phi(s_0 + i\varepsilon) = 0 \tag{1.13}$$

by the secant method for a sufficiently small positive value of ε.

The curvature of the path of steepest descent at the saddlepoint is given by[2]

$$\frac{1}{3}\,\Phi'''(s_0)/\Phi''(s_0).$$

When the variable V is close to the mean $\bar{V}$, however, the path of steepest descent may begin at $s = s_0$ to curve in one direction and then bend back and go off to infinity in the opposite direction before the value of the integrand has become sufficiently small so that the numerical integration can be stopped. This behavior can be avoided by taking the curvature of the parabolic contour of integration instead as

$$\kappa = \frac{1}{3}\,\tilde{\Phi}'''(s_0)/\tilde{\Phi}''(s_0), \tag{1.14}$$

where

$$\tilde{\Phi}(s) = \ell n\ h(s) + Vs \tag{1.15}$$

is the modified phase, which dominates the variation of the integrand far from the saddlepoint. Because the curvature κ is not needed to great accuracy, it often suffices to take it simply as

$$\kappa = \kappa_3/3\sigma^2, \tag{1.16}$$

where $\kappa_3 = E(V-\bar{V})^3$ is the third cumulant of V and $\sigma^2 = \text{Var } V$.

It sometimes happens that one knows the moments μ_n of the random variable V, but has no analytical form for the m.g.f. h(s). One can then express the phase $\Phi(s)$ as

$$\Phi(s) = \sum_{n=1}^{\infty} b_n(-s)^n + Vs - \ell n\ s, \tag{1.17}$$

whose coefficients $b_n = \kappa_n/n!$ can be determined from the quantities $a_n = \mu_n/n!$ by the recurrence in (1.7). One stops the summation in (1.17) when the term $b_n(-s)^n$

becomes sufficiently small. Alternatively one can take

$$\Phi(s) = \ln h(s) + Vs - \ln s$$

and compute h(s) from the power series in (1.2). In many problems the magnitude of the integrand of (1.4), (1.5), or (1.11) decreases to a level at which the integration can be terminated while the point $s = x + iy$ is still within the domain of convergence of the series in (1.2) and (1.17).

2. The Integrated Output of a Linear Rectifier

In the receiver of a radar searching for a target at a particular range, the output of a narrowband filter matched to the echo signal is rectified and sampled at an appropriate time to provide a datum v. The data $v_1, v_2, \ldots, v_M$ acquired during M successive interpulse intervals are summed, or "integrated," to yield a statistic

$$V = v_1 + v_2 + \ldots + v_M, \tag{2.1}$$

and if this exceeds a certain decision level V_0, the radar decides that a target is present.

When the noise is Gaussian and the rectifier has a quadratic characteristic, the detection probability $q^+(V_0) = \Pr(V > V_0)$ is given by the M-th-order Q-function.[3] We have previously treated the computation of this function by saddlepoint integration.[4] If the rectifier is linear, on the other hand, the distribution of V is not known in analytical form. Marcum[5] used the Edgeworth series to approximate the false-alarm and detection probabilities of a system integrating the outputs of a linear rectifier. This problem is conveniently handled by the method described in Section 1.

With appropriate normalization we can write

$$v_k = [(s_x + x_k)^2 + (s_y + y_k)^2]^{1/2} \tag{2.2}$$

where x_k and y_k are independent Gaussian random variables with mean zero and unit variance, and $S = \frac{1}{2}(s_x^2 + s_y^2)$ is the input signal-to-noise ratio, assumed the same for all k, $1 \leq k \leq M$. Then the probability density function of the v_k's is

$$p(v) = v \exp(-\tfrac{1}{2} v^2 - S)\, I_0(\sqrt{2S}v), \qquad v > 0, \tag{2.3}$$

where $I_0(\cdot)$ is the modified Bessel function (ref. 3, p. 168). The moments of v_k

were given by Rice (ref. 6, eq. (3.10-12), p. 107) as

$$\mu_n = E(v_k^n) = 2^{n/2}\Gamma(\tfrac{n}{2} + 1)\ {}_1F_1(-n/2,\ 1;\ -S) \tag{2.4}$$

in terms of the confluent hypergeometric function

$${}_1F_1(a,1;x) = 1 + \sum_{r=1}^{\infty} \frac{\Gamma(a+r)}{\Gamma(a)}\ \frac{x^r}{(r!)^2}. \tag{2.5}$$

When S = 0 the m.g.f. of v_k is given in closed form by[5]

$$h_0(s) = E[\exp(-v_k s)] = 1 - (2\pi)^{1/2}\ s\ e^{s^2/2}\ \text{erfc}\ s, \tag{2.6}$$

where

$$\text{erfc}\ s = (2\pi)^{-1/2} \int_s^{\infty} e^{-u^2/2}\ du \tag{2.7}$$

is the error-function integral. The m.g.f. of V is then

$$h(s) = [h_0(s)]^M. \tag{2.8}$$

When n is large, the n-th moment μ_n is asymptotically

$$\mu_n \cong 2^{n/2}\Gamma(\tfrac{n}{2} + 1)e^{-S/2}\ I_0(\sqrt{2(n+1)S})$$

by eq. (13.5.13), p. 508 of Abramowitz and Stegun.[7] Using Sterling's formula for the factorial and the asymptotic formula for the modified Bessel function, we find for $n \gg 1$

$$a_n = \mu_n/n! \sim (8\pi nS)^{-1/2}\ n^{-n/2}\ \exp[\tfrac{n-S}{2} + (2nS)^{1/2}].$$

These coefficients decrease so rapidly with increasing n that the power series in (1.2) for the m.g.f. of the v_k converges everywhere in the complex s-plane.

The first four coefficients of this series

$$h_0(s) = \sum_{k=0}^{\infty} a_k(-s)^k \tag{2.9}$$

are

$$a_0 = 1,\ a_1 = (\pi/2)^{1/2}\ {}_1F_1(-\tfrac{1}{2},1;-S)$$

$$a_2 = 1 + S,\ a_3 = (\pi/8)^{1/2}\ {}_1F_1(-3/2,1;-S). \tag{2.10}$$

The rest can be evaluated by the recurrence

$$a_{m+2} = \frac{2(m+1+S)(m-1)a_m - ma_{m-2}}{(m^2 - 1)(m + 2)} \tag{2.11}$$

so that the series in (2.5) needs to be summed only twice. One can then use the recurrence in (1.7) to determine the coefficients of the series

$$\ln h_0(s) = \sum_{k=0}^{\infty} b_k(-s)^k. \tag{2.12}$$

Multiplying these by M yields the terms in the expression (1.17) for the phase $\Phi(s)$ of the integrands of (1.4), (1.5), and (1.11).

The false-alarm probability $Q_0 = \Pr(V > V_0 | S = 0)$ can be computed by numerical quadrature of (1.11), using (2.6)-(2.8) to calculate h(s) on the path of integration. The saddlepoint s_0 was calculated by Newton's method as in (1.12), and the curvature κ of the parabolic path of integration was determined by (1.14). By combining this method with the secant method we were able to determine the decision level V_0 to attain a pre-assigned false-alarm probability Q_0. The detection probability $Q_d = \Pr(V > V_0 | S > 0)$ could then be calculated by (1.11), in which the m.g.f. h(s) was computed as in (2.9)-(2.12), with (2.8). This method worked very efficiently.

3. The Average Power of a Random Process

Methods for calculating the cumulative distribution of the average power

$$V = T^{-1} \int_0^T [z(t)]^2 \, dt$$

of a Gaussian random process $z(t)$ with a given mean $s(t)$ and a given auto-covariance function $\phi(t-s)$ were described in Sec. 3 of our 1982-83 annual report.[8] Research into these is continuing with emphasis on our adaptation of the Kreĭn-Levinson method in part (b) of that section. The accuracy of the method has been improved by computing the estimator kernel

$$A_u(\tau,r) = \tilde{A}_u(\tau,r) + u(1+\tau u)^{-1}$$

from an integral equation for $\tilde{A}_u(\tau,r)$ instead of directly from eq. (3.22) of that report, following the suggestion made on p. 20 thereof.[8] The new integral equation has been solved numerically by the method of conjugate gradients in combination with the Sobolev identity (eq. (3.25) of ref. 8). The solution enables more precise computation of the moment-generating function $h(u)$ of V at points u far out on the contour of integration in (1.11), and the results for the cumulative distribution of V are consequently more accurate.

4. Positive-Integer-Valued Random Variables

Let V be a random variable whose value can be only a positive integer or zero. If $p_k = \Pr(V = k)$, $k = 0, 1, 2, \ldots$, the probability generating function (p.g.f.) of V is defined by

$$h(z) = E(z^V) = \sum_{n=0}^{\infty} p_n z^n. \tag{4.1}$$

In terms of it the probabilities p_k are given by the contour integral

$$p_n = \int_C h(z)\ z^{-(n+1)}\ dz/2\pi i, \tag{4.2}$$

where C is a closed contour enclosing the origin, but no singularities of h(z), all of which lie outside the unit circle. The cumulative distribution of V is given by

$$q_k^- = \Pr(V < k) = \sum_{n=0}^{k-1} p_n = \int_{C_-} \frac{z^{-k}h(z)}{1 - z}\ \frac{dz}{2\pi i} \tag{4.3}$$

and its complement by

$$q_k^+ = \Pr(V > k) = \sum_{n=k}^{\infty} p_n = \int_{C_+} \frac{z^{-k}h(z)}{z - 1}\ \frac{dz}{2\pi i}. \tag{4.4}$$

Here C_- and C_+ are contours like C, except that C_+ encloses the point $z = 1$ and C_- does not.

As described in Section 4 of our 1982-83 annual report,[8] these integrals can often be expeditiously evaluated by deforming C_- and C_+ into contours passing through saddlepoints x_0^- and x_0^+ of their integrands and approximating the paths of steepest descent. The saddlepoints are roots of the equation

$$\Psi'(z) = \frac{h'(z)}{h(z)} - \frac{n}{z} - \frac{1}{z-1} = 0. \qquad (4.5)$$

where

$$\Psi(z) = \ell n\ h(z) - n\ \ell n\ z - \ell n[\pm(z-1)] \qquad (4.6)$$

is the "phase" of the integrand. Applications to the distribution of the number of electrons at the output of a photomultiplier were treated in that report. Further applications will now be outlined.

5. Photoelectron Counting

When light falls on a photoelectrically emissive surface, the probability generating function of the number of electrons ejected during an interval (0, T) is given by

$$h(z) = E[e^{I(z-1)}], \tag{5.1}$$

where I is a random variable proportional to the total energy falling on the surface during that interval. If we assume the light to be plane-polarized, quasimonochromatic, and spatially coherent over the entire emissive surface, its field can be written

$$v(t) = \mathrm{Re}\ [V(t)e^{i\Omega t}], \qquad 0 < t < T,$$

in which the complex envelope V(t) is a circular complex Gaussian random process, and $\Omega = 2\pi\nu$ is the central angular frequency of the light. Then

$$I = \frac{1}{2}\eta \int_0^T |V(t)|^2\, dt \tag{5.2}$$

with η a constant directly proportional to the quantum efficiency of the emissive surface and inversely proportional to the photon energy $h\nu$.

The expected value $E[V(t)] = S(t)$ represents the complex envelope of a coherent light signal, such as the output of a pulsed laser. The difference $V_b(t) = V(t) - S(t)$ represents incoherent light such as is produced by an ordinary incandescent source. It is assumed to be a stationary random process with a temporal coherence function defined by

$$G(t_1-t_2) = \frac{1}{2} E[V_b(t_1)V_b^*(t_2)]. \tag{5.3}$$

The spectral density of the light is

$$\Gamma(\omega) = \int_{-\infty}^{\infty} G(t) e^{i\omega t}\, dt. \tag{5.4}$$

The mean number n_0 of photoelectrons ejected by this incoherent component is

$$n_0 = \eta G(0) T; \tag{5.5}$$

the mean number n_s ejected by the coherent component is

$$n_s = \tfrac{1}{2}\eta \int_0^T |S(t)|^2\, dt. \tag{5.6}$$

The p.g.f. h(z) can be expressed as[9]

$$h(z) = [D(\xi)]^{-1} \exp\left[n_s(z-1) \int_0^T \int_0^T \sigma^*(t)\psi(t,u;\xi;T)\sigma(u)\, dtdu\right],$$

$$\xi = n_0(1-z), \tag{5.7}$$

where

$$D(\xi) = \prod_{k=1}^{\infty} (1+\lambda_k \xi) \tag{5.8}$$

is the Fredholm determinant associated with the integral equation

$$\lambda_k f_k(t) = T^{-1} \int_0^T \gamma(t-s)\, f_k(s)\, ds, \quad \gamma(\tau) = G(\tau)/G(0), \tag{5.9}$$

whose eigenvalues are λ_k. Furthermore, $\sigma(t)$ is a normalized signal,

$$\sigma(t) = S(t) \Big/ \int_0^T |S(u)|^2\, du, \tag{5.10}$$

and $\psi(t,u;\xi;T)$ is the resolvent kernel defined by the integral equation

$$\psi(t,u;\xi;\tau) + \frac{\xi}{T}\int_0^\tau \psi(t,v;\xi;\tau)\gamma(v-u)\,dv$$

$$= \xi T^{-1}\gamma(t-u), \qquad 0 \leq (t,u) \leq \tau \leq T. \tag{5.11}$$

In the past the distribution of the number of photoelectrons has been evaluated by decomposing the field V(t) into temporal modes by a Karhunen-Loeve expansion, whereupon the p.g.f. can be written as a product of factors involving the eigenvalues λ_k of (5.9) and the coefficients

$$S_k = \int_0^T f_k^*(t)\, S(t)\, dt$$

of the coherent component. The distribution corresponding to each factor or mode has the Laguerre form, and the distribution of the total number V of photoelectron counts is the convolution of as many of these as necessary.[10] Other modal decompositions can also be used.[11] (Further references are to be found in ref. 9.) Such methods require much computation time and storage, particularly when the product WT of the counting time T and the bandwidth W of $\Gamma(\omega)$ is large. The method of saddlepoint integration promises much greater efficiency in calculating the counting distribution.

When the light has a Lorentz spectral density,

$$\Gamma(\omega) = \frac{2\mu G(0)}{\omega^2+\mu^2}, \qquad \gamma(\tau) = e^{-\mu|\tau|}, \tag{5.12}$$

the Fredholm determinant has the well-known form[12]

$$D(\xi) = e^{-m}[(g+1)^2 e^{mg} - (g-1)^2 e^{-mg}]/(4g),$$

$$g = (1+2\xi/m)^{1/2}, \quad \xi = n_0(1-z), \quad m = \mu T. \tag{5.13}$$

If the signal is a sinusoid of constant amplitude and with a frequency displaced

from Ω by Δ, as we have shown,[9] the p.g.f. $h(z)$ is

$$h(z) = [D(\xi)]^{-1} \exp\{n_s(z-1)[\frac{1+\delta^2}{g^2+\delta^2} + \frac{4\xi\{(g+1)(g-\delta^2)e^{mg} - (g-1)(g+\delta^2)e^{-mg} - 2g[(1-\delta^2)\cos m\delta - 2\delta \sin m\delta]\}}{(g+1)^2 e^{mg} - (g-1)^2 e^{-mg}]\}} \tag{5.14}$$

with $\delta = \Delta/\mu$ and the rest of the notation as in (5.13). After putting this into the integrals (4.3) and (4.4) we have computed the complementary cumulative distribution q_k^+ by evaluating the integrals numerically along a parabolic contour passing through the appropriate saddlepoint. Even for values of k of the order of thousands, accuracy of seven significant figures was attained by fewer than twenty steps of numerical integration.

For more complicated spectra $\Gamma(\omega)$ the p.g.f. $h(z)$ cannot be written down in closed form. It can be obtained numerically at points on the contour of integration, however, from the alternative expression[9]

$$h(z) = [D[\xi)]^{-1} \times \exp\{n_s(z-1)\int_0^T [\tilde{\sigma}^*(\tau;\xi^*)-\sigma^*(\tau)][\tilde{\sigma}(\tau;\xi)-\sigma(\tau)]d\tau\} \tag{5.15}$$

where

$$\tilde{\sigma}(\tau;\xi) = \int_0^\tau \bar{\psi}(\tau,u;\xi)\sigma(u)du \tag{5.16}$$

and $\bar{\psi}(\tau;u;\xi) = \psi(\tau,u;\xi;\tau)$ is the solution of the integral equation

$$\bar{\psi}(\tau,u;\xi) + \xi T^{-1}\int_0^{\tau} \bar{\psi}(\tau,v;\xi)\gamma(v-u)dv$$

$$= \xi T^{-1}\gamma(\tau-u), \qquad 0 \leq u \leq \tau \leq T. \tag{5.17}$$

In terms of this function the Fredholm determinant is given by[12]

$$D(\xi) = \exp\Big[\int_0^T \bar{\psi}(\tau,\tau;\xi)d\tau\Big]. \tag{5.18}$$

The function $\bar{\psi}(\tau,u;\xi)$ is the kernel of the minimum-mean-square-error causal estimator of the process $(\xi/T)^{1/2}V_b(\tau)$ in the presence of circular-complex white Gaussian noise with unit spectral density. As such it can be computed by solving Kalman-Bucy or Chandrasekhar differential equations whenever one can establish a state-space model for generating $V_b(t)$ from white noise. Indeed, the procedures for determining the distribution of the average power of a Gaussian random process, described in Sec. 3 of our 1982-83 annual report,[8] can be applied to evaluating the terms in h(z) in (5.15). In particular, one can solve the integral equation (5.17) numerically by sampling in τ and u, replacing the integral by a quadrature formula, and solving the resulting linear simultaneous equations by the method of conjugate gradients as outlined there.[8]

When the product WT of the counting time T and the bandwidth W of the incoherent light is large, one can utilize the so-called "Toeplitz approximation" to write the p.g.f. as[9]

$$h(z) \cong \exp\Big\{n_s(z-1)\int_{-\infty}^{\infty} \frac{|\Sigma(\omega)|^2}{1+\xi T^{-1}\tilde{\Gamma}(\omega)}\,\frac{d\omega}{2\pi} - T\int_{-\infty}^{\infty} \ell n[1+\xi T^{-1}\tilde{\Gamma}(\omega)]d\omega/2\pi\Big\},$$

$$\tilde{\Gamma}(\omega) = \Gamma(\omega)/G(0), \quad \Sigma(\omega) = \int_{-\infty}^{\infty}\sigma(t)e^{-i\omega t}dt, \quad \xi = n_0(1-z). \tag{5.19}$$

This can be substituted into (4.3) or (4.4) for numerical integration to

determine the cumulative distribution of the number of photoelectrons approximately. The result has been found quite close to the true distribution yielded by (5.14) for a time-bandwidth product $m = \mu T$ as low as 5.

6. Distribution of the Output of an Avalanche Photodiode

In an avalanche photodiode primary electrons generated at one face by incident light are accelerated under an applied voltage and create hole-electron pairs in the body of the diode by collision. The holes and the secondary electrons are themselves accelerated and create more such pairs. The probability Pr (n| N) that a fixed number N of primary electrons generate a total of n electrons at the output was calculated by McIntyre[13] under the assumption that the collision-ionization probability for holes is a fixed fraction g of that for electrons, independently of their energy, $0 < g < 1$. Personick[14] showed under the same assumption that the probability generating function (p.g.f)

$$M(z) = \sum_{n=0}^{\infty} \Pr\ (n|\ 1)\ z^n \tag{6.1}$$

of the distribution of the number of output electrons arising from a single primary electron is the solution of the equation

$$z = M[1 + a(M-1)]^{-b}, \qquad b = (1-g)^{-1} > 1, \tag{6.2}$$

where a is related to the gain G of the diode by

$$G = (1-ab)^{-1}, \qquad 0 < a < 1. \tag{6.3}$$

Ordinarily the number of primary electrons has a Poisson distribution with mean λ proportional to the intensity of the incident light, and the number n of electrons at the output has a compound Poisson distribution,

$$p_n = \sum_{N=0}^{\infty} \frac{\lambda^N e^{-\lambda}}{N!} \Pr\ (n|\ N). \tag{6.4}$$

The cumulative distribution, that is, the probability q_k^- that fewer than k

electrons are counted, must then be computed by summing the probabilities p_n,

$$q_k^- = \sum_{n=0}^{k-1} p_n, \tag{6.5}$$

and the complementary cumulative distribution is $q_k^+ = 1 - q_k^-$. Computing these probabilities in this manner is tedious because of the complexity of McIntyre's formula[13] for Pr (n| N), and when, as usually in a multiplicative device such as this, the numbers k are large, many summations are required and round-off error is serious.

The method of saddlepoint integration has been found most efficient for determining the cumulative distribution q_k^- of the number of output electrons.[15] One computes q_k^- or q_k^+ by numerical integration of (4.3) or (4.4), in which now

$$h(z) = \exp[\lambda(M-1)], \tag{6.6}$$

M and z being related by (6.2). Rather than solving (6.2) for M at each point z on the contour of integration, we change variables in (4.3) and (4.4) and use M as the variable of integration, obtaining the formula

$$q_k^{\pm} = \pm \int_{C_\pm} e^{\lambda(M-1)}\, M^{-n} [1 + a(M-1)]^{bn-1} \times \left[\frac{1 - a + a(1-b)M}{M - [1 + a(M-1)]^b}\right] \frac{dM}{2\pi i}, \tag{6.7}$$

where C_- and C_+ are curves in the M-plane passing through the image M_0^- or M_0^+ of the saddlepoint x_0^- or x_0^+ of the integrand in (4.3) or (4.4). The M-plane is cut along the real M-axis from a point

$$M_* = (1-a)/[a(b-1)] > 1$$

to $M = \infty$, corresponding to a cut in the z-plane along the Re z-axis from a point x_* to ∞, in order to render the solution of (6.2) for M single-valued. A parabolic contour in the cut M-plane was found to yield $q_k^{\pm}$ to seven significant figures with fewer than twenty steps of numerical integration, even for values of k as high as 50000. Indeed, the larger the gain G and the mean number λ of primary electrons, the more accurate this method of saddlepoint integration becomes.

In an optical receiver utilizing an avalanche photodiode, Gaussian noise is added to the diode output in a subsequent amplifier. The performance of such a receiver was calculated by Personick, Balaban, Bobsin, and Kumar[16] by convolving the distribution of the number of output electrons, as computed by McIntyre's formula,[13] with a Gaussian distribution representing the noise. This computation was so lengthy that they resorted instead to Monte Carlo simulation and a crude Gaussian approximation. This problem should yield easily to the method of saddlepoint integration. The moment-generating function (m.g.f.) of the sum V of the number of output electrons and Gaussian noise with mean zero and variance σ^2 is

$$h(s) = E[e^{-sV}] = \exp[\lambda(M-1) + \frac{1}{2}\sigma^2 s^2], \tag{6.8}$$

with $s = e^{-z}$ and z related to M again through (6.2). The cumulative distribution of the noisy output V and its complement can be computed by numerical integration of (1.4) and (1.5). Once again we expect to achieve this by using M instead of s as the variable of integration.

Extending a previous analysis,[17] we have applied the saddlepoint approximation[18] to this problem, evaluating the probabilities $q^-(V)$ and $q^+(V)$ as

$$q^{\pm}(V) \cong [2\pi\Phi''(s_0)]^{-1/2} \exp \Phi(s_0), \tag{6.9}$$

where the phase $\Phi(s)$ is, as in (1.10),

$$\Phi(s) = \lambda(M-1) + \frac{1}{2}\sigma^2 s^2 + Vs - \ell n\ (\pm s), \tag{6.10}$$

and the saddlepoint s_0 is the root of

$$\Phi'(s) = \lambda \frac{dM}{ds} + \sigma^2 s + V - \frac{1}{s} = 0 \tag{6.11}$$

lying to the left of the origin for $q^+(V)$ and to the right of the origin for $q^-(V)$. By using M as the independent variable, (6.11) could be solved by Newton's method.

In the binary receiver analyzed by Personick et al.,[16] the decision level V_0 above which it decides that a signal is present is set to minimize the average probability of error. As 0's and 1's were taken as equally likely, this is the point at which $p_0(V) = p_1(V)$, where $p_0(V)$ and $p_i(V)$ are the probability density functions of V under the hypotheses H_0 (no signal) and H_1 (signal present), respectively. These density functions can also be calculated approximately by the saddlepoint method[18];

$$p_0(V) \cong h_0(\tilde{s}_0)\ \exp(\tilde{s}_0 V)[2\pi\tilde{\Phi}_0''(\tilde{s}_0)]^{-1/2},$$

$$p_1(V) \cong h_1(\tilde{s}_1)\ \exp(\tilde{s}_1 V)[2\pi\tilde{\Phi}_1''(\tilde{s}_1)]^{-1/2}, \tag{6.12}$$

where $h_i(s)$ is the m.g.f. of V under hypothesis H_i and

$$\tilde{\Phi}_i(s) = \ell n\ h_i(s) + sV, \qquad i = 0,\ 1,$$

is the associated modified phase. The saddlepoints $\tilde{s}_0$ and $\tilde{s}_1$ are the roots of the equations

$$\tilde{\Phi}_i'(s) = \frac{d}{ds} h_i(\varepsilon) + V = 0, \qquad i = 0,\ 1, \tag{6.13}$$

which can be solved by a few iterations of Newton's method. One then solves the equation

$$\ln p_1(V_0) - \ln p_0(V_0) = 0$$

for V_0 by the secant method, starting at an initial trial value halfway between the means $G\lambda_0$ and $G\lambda_1$ of the output V under the two hypotheses. Personick *et al.*[16] determined the mean number λ_1 of primary electrons needed to attain an average error probability of 10^{-9}. For this one applies the secant method taking that mean number as the independent variable. By using the saddlepoint approximation in (6.9)-(6.13) we were able to reproduce the results plotted in ref. 16 in about half an hour's computation on an HP-71 portable computer.

7. Significance Probabilities for Rank Tests

The method of saddlepoint integration described in Sec. 4 has been applied to computing the cumulative distributions of the Wilcoxon signed-rank and rank-sum statistics under the conventional null hypothesis.[19] Given M data $x_1, x_2, \ldots, x_M$, the signed-rank statistic is

$$W = \sum_{i=1}^{M} r_i \, U(x_i), \tag{7.1}$$

where r_i is the rank of the i-th datum when the x's are arranged in order of their absolute values, and $U(\cdot)$ is the unit step-function. Given two sets of data $x_1, x_2, \ldots, x_M$ and $y_1, y_2, \ldots, y_N$, the rank-sum statistic is

$$V = \sum_{i=1}^{M} r_i, \tag{7.2}$$

where r_i is the rank of x_i when all the M+N data — x's and y's — are arranged in ascending order. It is convenient to define the statistic

$$U = NM + \tfrac{1}{2} M(M+1) - V, \tag{7.3}$$

which equals the number of times an x precedes a y in a random arrangement of the M x's and N y's according to their values. Under the null hypothesis they are all independent and identically distributed. These statistics form the basis of frequently used statistical hypothesis tests known as "rank tests."

Table of the cumulative distributions of W and V or U and tables of the percentage points of these distributions, used in assessing the significance of the outcome of such a rank test, take up a great many pages, even for values of M and N up to 50.[20] Although recurrent relations exist whereby the distributions can be computed, the algorithms are extremely time-consuming and require a large

amount of computer memory when M and N are large. One would like to have a means of computing the cumulative probability q_k^- or its complement for the integer-valued random variables W and V (or U) directly.

The p.g.f.'s of these statistics under the null hypothesis are[20]

$$h_W(z) = E(z^W) = \prod_{j=1}^{M} \left(\frac{1+z^j}{2}\right) \tag{7.4}$$

and

$$h_U(z) = E(z^U) = c \prod_{j=1}^{M} \left(\frac{z^{N+j-1}}{z^{j-1}}\right), \quad c = \binom{N+M}{M}^{-1}. \tag{7.5}$$

Because of the symmetry of the distributions of W and U about their means,

$$E(W) = M(M+1)/4, \qquad E(U) = MN/2, \tag{7.6}$$

it suffices to compute the probabilities Pr (W ≤ k) and Pr (U ≤ k) for integers k up to the mean value of the statistic, and this can be done by numerical integration of

$$\Pr\ (W \leq k) = \int_{C_-} \frac{z^{-(k+1)}\ h_W(z)}{1 - z}\ \frac{dz}{2\pi i} \tag{7.7}$$

with a similar formula for Pr (U ≤ k). For the latter, no generality is lost by taking M ≤ N. The integrand has a single saddlepoint x_0 on the Re z-axis in $0 < \text{Re } z < 1$. The contour C was taken as a vertical straight line through $z = x_0$, completed by the portion of the unit circle lying to the left of it. Extensive numerical comparison with the exact probabilities showed that for $M \gtrsim 30$ the contribution of the circular part of the contour C to the integral is negligible, and it suffices to approximate the probability in (7.7) by

$$\Pr(W \le k) \cong \mathrm{Re}\int_0^Y \frac{z^{-(k+1)}\, h_W(z)}{1-z}\,\frac{dy}{\pi}, \quad Y = (1-x_0^2)^{1/2}, \quad z = x_0 + iy. \qquad (7.8)$$

The integrand of (7.8) drops off rapidly as y increases from 0, and for M of the order of 50 or more it is unnecessary to carry the integration all the way to the unit circle (y = Y). The probability Pr (U ≤ k) is computed in the same manner.

The amount of memory required by this method -- in contrast to algorithms previously used -- is independent of M and N. It was even possible to program it on an HP-15C calculator. The amount of computation time is roughly proportional to M because of the need to compute the p.g.f. in (7.4) or (7.5) at each point on the contour of integration. This saddlepoint-integration method enables the significance probability of a set of ranked data to be computed directly and eliminates the need for cumbersome tables of distributions and percentage points.

8. Robust Radar Detectors (J. A. Ritcey)

Mean-level detectors (MLD) are commonly used in radar to maintain a constant false-alarm rate (CFAR) when the background noise level is unknown. In our annual report for 1982-83 we reported on the application of saddlepoint integration techniques to radar detection problems where the test statistic is compared to either a fixed threshold or to an adaptive threshold composed of a weighted estimate of the mean level of the background noise, derived from a given reference channel.[4] The method was developed for an arbitrary number of pulses noncoherently integrated and for both a nonfluctuating and a chi-square fluctuating target. Recently, we have considered a variant of the basic MLD that is more robust in the presence of interference in the reference channel, and we have obtained exact analytical results for the probabilities of detection and false alarm when the interference-to-noise ratio (INR) is nonzero. In practice, the interference could result either from returns from real objects such as other targets or clutter, or from pulsed-noise jamming. Here we consider only Swerling II target returns and only single-pulse processing (no noncoherent integration). The approach is to write the detection probability as a contour integral, which is evaluated by the method of residues. Variants of this system in which noncoherent integration is employed or other target models are considered would require numerical contour integrations. In the model we consider here, developing an analytical expression for the moment-generating function (m.g.f.) of the reference-channel estimate is the primary mathematical problem.

Radar detection of a known signal in additive white Gaussian noise of unknown variance is often accomplished by comparing the test statistic for a single range cell with an adaptive threshold equal to a scaled estimate of the unknown noise variance. For a system that quadratically rectifies the output of

a matched filter to obtain the test statistic, the problem can be modeled by the following hypothesis testing problem:

$$H_0: \quad v = \frac{1}{2}(x^2 + y^2)$$

$$H_1: \quad v = \frac{1}{2}\left((x+s)^2 + (y+t)^2\right); \ |d|^2 = s^2 + t^2. \tag{8.1}$$

The random variables x, y are independent Gaussian noise samples with zero mean and unit variance. The target echo provides a fixed signal-to-noise ratio (SNR) for a steady target, and the amplitude can be taken as a random variable when the echo fluctuates. The phase of the complex amplitude d = s + it is uniformly distributed over (0, 2π), and the total SNR is

$$S = |d|^2/2. \tag{8.2}$$

Implementing the generalized likelihood ratio test, the system decides for H_0 or H_1 according to whether

$$v \underset{H_0}{\overset{H_1}{\gtrless}} ar \tag{8.3}$$

where r is an estimate of the noise variance and a is a positive threshold parameter. Typically the radar uses the n range cells surrounding the cell under test to compute an estimate

$$r = \sum_{j=1}^{n} z_j \tag{8.4}$$

where

$$z_j = \frac{1}{2}(x_j^2 + y_j^2) \tag{8.5}$$

and the x_j, y_j j = 1,2, ..., n are independent and identically distributed

(i.i.d.) zero-mean Gaussian random variables, and we have incorporated a scaling factor n^{-1} into the threshold parameter a. In a locally homogeneous noise environment the x_j, y_j all have common unit variance. The performance is completely specified by the false alarm and detection probabilities

$$Q_0 = \int_0^\infty f(r) \int_{ar}^\infty p_0(v)\, dv\, dr \tag{8.6}$$

$$Q_d = \int_0^\infty f(r) \int_{ar}^\infty p_1(v)\, dv\, dr \tag{8.7}$$

where f(r) is the probability density function (p.d.f.) of the estimate r and $p_i(v)$, i = 0, 1, is the p.d.f. of v under each hypothesis. An equivalent hypothesis test is

$$z = v - ar \underset{H_0}{\overset{H_1}{\gtrless}} 0 \tag{8.8}$$

where the random variable z has a p.d.f. $p_z(z)$ whose Laplace transform is given by

$$h_z(u) = \int_0^\infty e^{-uz}\, p_z(z)dz = \int_0^\infty e^{-uv}\, p(v)dv \int_0^\infty e^{aru}\, f(r)dr \tag{8.9}$$

or

$$h_z(u) = \langle e^{-uz} \rangle = \langle e^{-uv} \rangle \langle e^{uar} \rangle = h(u)\, d(-au) \tag{8.10}$$

where h(u), d(u) are the moment-generating functions (m.g.f.) of v and r respectively. If we specify a Swerling II target model, the m.g.f. is given by

$$h(u) = (1+bu)^{-1};\ b = 1 + \bar{S} \tag{8.11}$$

and $\bar{S} = \frac{1}{2} \langle |d|^2 \rangle$ is the average SNR. The estimate r, in a homogeneous environment, has the m.g.f. d(u) given by

$$d(u) = (1+u)^{-1} \tag{8.12}$$

The detection performance can be written as contour integrals involving these m.g.f.'s. Since $h_z(u)$ and $p_z(z)$ are related by an inverse Laplace transform, we have directly from (8.8) that

$$Q_d = \Pr [z > 0 \mid H_1] = \int_0^\infty \oint \frac{du}{2\pi i} e^{uz} h(u) \, d(-au) dz \tag{8.13}$$

or

$$Q_d = - \oint_{C_-} \frac{du}{2\pi i} u^{-1} h(u) \, d(-au) \tag{8.14}$$

where the contour of integration C_- consists of a vertical path in the complex u-plane crossing the negative real axis at $u = c_1$, $b^{-1} < c_1 < 0$. The contour is closed in an infinite semi-circle in the left half-plane (LHP). Since d(u) is the m.g.f. of a positive random variable, the only singularities lie in the LHP. Therefore, d(-au) is analytic in the LHP and (8.14) can be immediately evaluated in terms of the residue at the simple pole, $u = -b^{-1}$. Thus, the detection probability is simply related to the m.g.f. of the estimate r by

$$Q_d = d(a/b); \; b = 1 + \bar{S}, \tag{8.15}$$

while

$$Q_0 = d(a).$$

The key point of this result is that given a closed-form analytical expression for the m.g.f. d(u) we immediately have a result for the detection performance.

Therefore, we are encouraged to investigate other systems that develop a different estimate r, but whose m.g.f. can still be obtained.

To reduce the degradation caused by interfering targets falling within the reference channel, Rickard and Dillard[21] proposed a class of detectors $\mathcal{D}_\ell$, $\ell = 0, 1, 2, \ldots, n-1$, where the ℓ largest range cells are censored from the noise-variance estimate. If the quadratically rectified output for the j-th range cell is denoted by z_j, the detector $\mathcal{D}_k$ uses the unscaled estimate

$$r = \sum_{j=1}^{k} z_{(j)} \tag{8.16}$$

where $k = n-1$ and $0 \leq z_{(1)} \leq z_{(2)} \leq \ldots \leq z_{(n)} \leq \infty$ are the order statistics of the sample $(z_1, z_2, \ldots, z_n)$. Since r is a biased estimator of the unknown noise variance we consider the more general statistic

$$r = \sum_{j=1}^{k-1} z_{(j)} + cz_{(k)} \tag{8.17}$$

for some constant $c > 0$. The contour integral (8.14) requires the m.g.f. of r, d(u).

For a homogeneous noise environment where the z_j's are i.i.d. with p.d.f. $p(z) = e^{-z}$, $z > 0$, the m.g.f. of r can be shown to be given by

$$d(u) = \langle e^{-ur} \rangle = \sum_{j=1}^{k} \left[1 + \left(\frac{c+k-j}{n+1-j}\right) u\right]^{-1}. \tag{8.18}$$

From (8.18) we note that the noise level estimate will be unbiased if we scale r by k^{-1} and choose $c = n + 1 - k$. Then

$$\frac{r}{k} = \frac{1}{k}\left[\sum_{j=1}^{k-1} z_{(j)} + (n+1-k) z_{(k)}\right] \tag{8.19}$$

is an unbiased estimate of the unknown noise variance that is robust with respect to the presence of several large interference pulses in the reference samples $(z_1, \ldots, z_n)$.

The performance of the CMLD in a multiple-target environment can be obtained directly from (8.15), given the m.g.f. d(u). It can be shown that the joint m.g.f.

$$h_\ell(u,s) = \langle \exp[-u(z_{(1)} + \ldots + z_{(k-1)}) - sz_{(k)}] \rangle, \tag{8.20}$$

where $z_{(1)} \leq z_{(2)} \leq \ldots \leq z_{(k)} \leq \ldots \leq z_{(n)}$ are the order statistics drawn from an independent and exponentially distributed sample $(z_1, z_2, \ldots, z_\ell \ldots, z_n)$ and where

$$\langle z_i \rangle = \begin{cases} \alpha^{-1} & i = 1, \ldots, \ell \\ 1 & i = \ell+1, \ldots, n \end{cases}$$

$$\alpha^{-1} = 1 + \text{INR} \tag{8.21}$$

is given by

$$h_\ell(u,s) = \binom{m}{\ell}^{-1} \sum_{i=0}^{\ell} \left\{ \binom{n-k}{i} \frac{\alpha^{\ell-1}}{(\ell-i)!} \right\}$$

$$\cdot \sum_{\substack{\ell_1=1 \\ \ell_1 \neq \ell_2 \neq \ldots \neq \ell_{\ell-i}}}^{k} \sum_{\ell_2=1}^{k} \cdots \sum_{\ell_{\ell-i}=1}^{k} \prod_{j=1}^{k} \left[1 + \frac{(\alpha-1)(i+r_j(i)) + \theta_j}{\psi_j} \right]^{-1} \tag{8.22}$$

where

$$\psi_j = n + 1 - j$$

$$\theta_j = s + u\,(k - j)$$

and

$$r_j(i) = \begin{cases} \gamma_j(\ell_1) + \ldots + \gamma_j(\ell_{\ell-i}) & i = 1, \ldots, \ell-1 \\ 0 & i = \ell \end{cases}$$

$$\gamma_j(\ell) = \begin{cases} 1 & j \leq \ell \\ 0 & j > \ell. \end{cases}$$

The multiple sum in the second line of (8.22) is present only when $i < \ell$. The reason we have evaluated the joint m.g.f. $h_\ell(u,s)$ is twofold: First, the CMLD with arbitrary bias constant c has a reference channel m.g.f. given by $h_\ell(u,cu)$ when ℓ interferers are present. Second, we can evaluate the performance of the so-called Order Statistic Detector (OSD) described by Rohling[22], which was the estimate $r = z_{(k)}$, a single order statistic. This system may be considerably easier to implement than the CMLD.

As an example of a particular result, Figure 8.1 shows the additional SNR required (often called the CFAR loss) for the MLD, CMLD, and OSD in a multiple target environment to achieve a given Q_d at fixed Q_0 over that required by the Neyman-Pearson detector, which is the optimal detector when the background noise level is known. Notice that although the CFAR loss increases with INR for the MLD, the CMLD and OSD have a bounded loss as the INR $\to \infty$. These results will be more fully reported in a paper in preparation for the IEEE-AES.

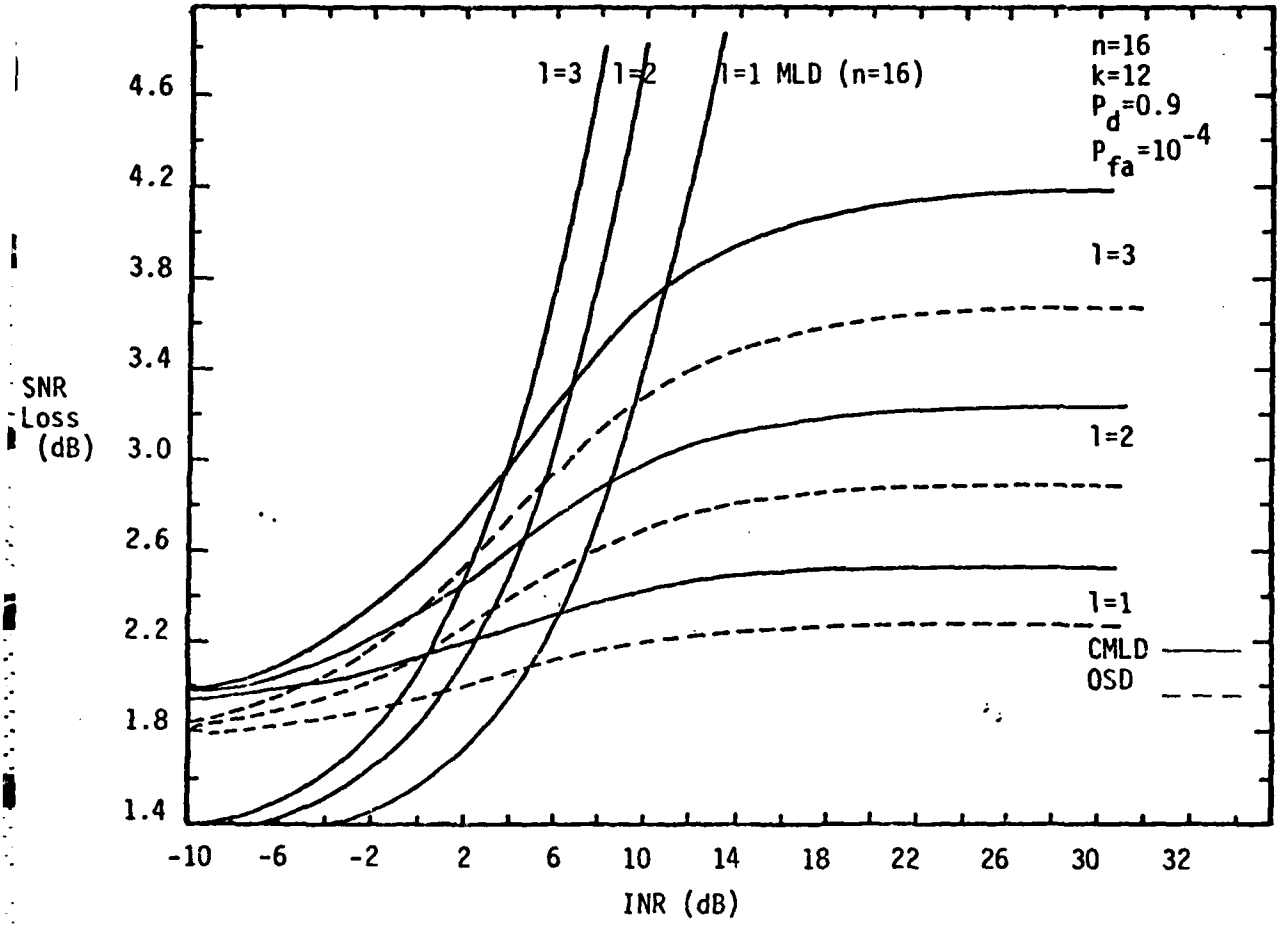

Figure 8.1 Comparison of the MLD, CMLD, and OSD in a multiple target environment.

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